Variable Structure and adaptive control of Nonlinear Systems

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15 *(a) State variables and (b) state plane, multi-hyperplane* ($a_1 = -1$, $a_2 = -1$, $a_3 = -1.8$, $C = 2$, $K = 3$, $r^2 = 100$, $\gamma_1 = \gamma_2 = \gamma_3 = 3$) $t_{CPU} = 12.5300\,\text{sec}$, $\sigma_v = 1$, *dither signal has sampling time* 0.01 *seconds, hyperplanes switching is related to* $e^{-t}(x_1^2 + x_2^2)$ *initial conditions used were* $(x_1, x_2) = (3, -1)$ 13

16 *(a) State variables and (b) state plane, multi-hyperplane* ($a_1 = -1$, $a_2 = -1$, $a_3 = -1.8$, $C = 2$, $K = 3$, $r^2 = 100$, $\gamma_1 = \gamma_2 = \gamma_3 = 3$) $t_{CPU} = 0.3900\,\text{sec}$, *hyperplanes switching is related to* $\frac{1}{(t+2)}(x_1^2 + x_2^2)$ *initial conditions used were* $(x_1, x_2) = (3, -1)$. 14

17 *(a) State variables and (b) state plane, multi-hyperplane* ($a_1 = -1$, $a_2 = -1$, $a_3 = -1.8$, $C = 2$, $K = 3$, $r^2 = 100$, $\gamma_1 = \gamma_2 = \gamma_3 = 3$) $t_{CPU} = 1.9600\,\text{sec}$, *hyperplanes switching is related to* $\frac{1}{(t+2)}(x_1^2 + x_2^2)$, $\sigma_v = 0.02$, *sampling time of the dither signal is* 0.01 *seconds, initial conditions used were* $(x_1, x_2) = (3, -1)$ 15

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Introduction

Aspects of designing centering around Variable Structure control of nonlinear systems was considered.

Problem Formulation

The first problem is

Problem 1 *With systems written in the form*

$$\dot{x} = f\left(x, u(x)\right), \tag{1}$$

where x is a vector and $u = -K\,\mathrm{sgn(s)},\ \mathrm{s} = \mathrm{C\,x_1 + x_2},\ \mathrm{K, C} \in \mathbb{R},\ \mathrm{K} > 0$, how to determine K and C.

The second problem is

Problem 2 *With systems in the form*

$$\dot{x} = f\left(a, x, u(x)\right), \tag{2}$$

where a is a vector of unknown parameters, how to design an adaptive control for this system.

Design Example

Referring to Problem 1 consider a system represented by

$$\dot{x_1} = x_2 \tag{3}$$
$$\dot{x_2} = a_1 x_1 + a_2 x_2 + a_3 \tan^{-1}\left(x_1 + x_2\right) + u. \tag{4}$$

Choose an input as

$$u = -K\,\mathrm{sgn(s)}, \quad \mathrm{s} = \mathrm{C\,x_1 + x_2}. \tag{5}$$

Then on the surface of the hyperplane we have $C\,x_1 + x_2 = 0$. And since $\dot{x_1} = x_2 = -C\,x_1$ we have

$$C > 0 \tag{6}$$

as a necessary condition for stability. Also the greater C is the quicker x_1 will goto the equilibrium point. Let us choose $C = 3$ for an instance.

Choose the Lyapunov function as

$$V = \frac{1}{2}s^2. \tag{7}$$

Then for stability

$$\begin{aligned} \dot{V} = s\dot{s} &= (C\,x_1 + x_2)\left[(1+a_2)x_2 + a_1x_1 + a_3\tan^{-1}(x_1+x_2) - K\,\mathrm{sgn}(\mathrm{C\,x_1 + x_2})\right] && (8)\\ &= a_1C\,x_1^2 + a_1x_1x_2 + C\,x_1x_2 + a_2C\,x_1x_2 + x_2^2 + a_2x_2^2 + a_3C\,x_1\tan^{-1}(x_1+x_2) + \\ &\quad a_3x_2\tan^{-1}(x_1+x_2) - C\,K\,x_1\mathrm{sgn}(\mathrm{C\,x_1 + x_2}) - \mathrm{K\,x_2sgn}(\mathrm{C\,x_1 + x_2}) \\ &< 0. && (9) \end{aligned}$$

To find K carry on with the computation.

$$\begin{aligned} a_1C\,x_1^2 + (1+a_2)\,x_2^2 + (a_1 + C + a_2C)\,x_1x_2 + \\ a_3\,(C\,x_1 + x_2)\tan^{-1}(x_1+x_2) - K(C\,x_1+x_2)\mathrm{sgn}(\mathrm{C\,x_1 + x_2}) \;<\; 0. && (10)\\ && (11) \end{aligned}$$

Notice that $K(C\,x_1 + x_2)\mathrm{sgn}(\mathrm{C\,x_1 + x_2}) = \mathrm{K}\,|\mathrm{C\,x_1 + x_2}|$. If $f\,(x_1, x_2) = a_1C\,x_1^2 + (1+a_2)\,x_2^2 + (a_1 + C + a_2C)\,x_1x_2 - a_3\,(C\,x_1 + x_2)\tan^{-1}(x_1+x_2)$ then

$$\begin{aligned} \frac{\partial f\,(x_1,x_2)}{\partial x_1} &= 2a_1Cx_1 + (a_1 + C + a_2C)x_2 + \frac{a_3(Cx_1+x_2)}{1+(x_1+x_2)^2} + a_3C\tan^{-1}(x_1+x_2) && (12)\\ \frac{\partial f\,(x_1,x_2)}{\partial x_2} &= (a_1 + C + a_2C)x_1 + 2(1+a_2) + \frac{a_3(Cx_1+x_2)}{1+(x_1+x_2)^2} + a_3\tan^{-1}(x_1+x_2). && (13) \end{aligned}$$

It would have been difficult to try to solve the equations

$$\begin{aligned} \frac{\partial f\,(x_1,x_2)}{\partial x_1} &= 0 && (14)\\ \frac{\partial f\,(x_1,x_2)}{\partial x_2} &= 0 && (15) \end{aligned}$$

therefore let us look at the graph of the Lyapunov function instead.

Figure 1 ($a_1 = -1, a_2 = -1, a_3 = -1.8, C = 2, K = 3$) shows an example of cases where Equation 10 is true and Figure 3 ($a_1 = 1, a_2 = 1, a_3 = -1.8, C = 2, K = 3$) shows the case when it is not. In Figure 1 $\max(v) = 0$ at the origin and in Figure 3 $\max(v) = 727.87$ at $(10, 10)$. The response of the systems with Lypunov functions as shown in Figure 1 and Figure 3 are shown in Figure 2 and Figure 4 respectively.

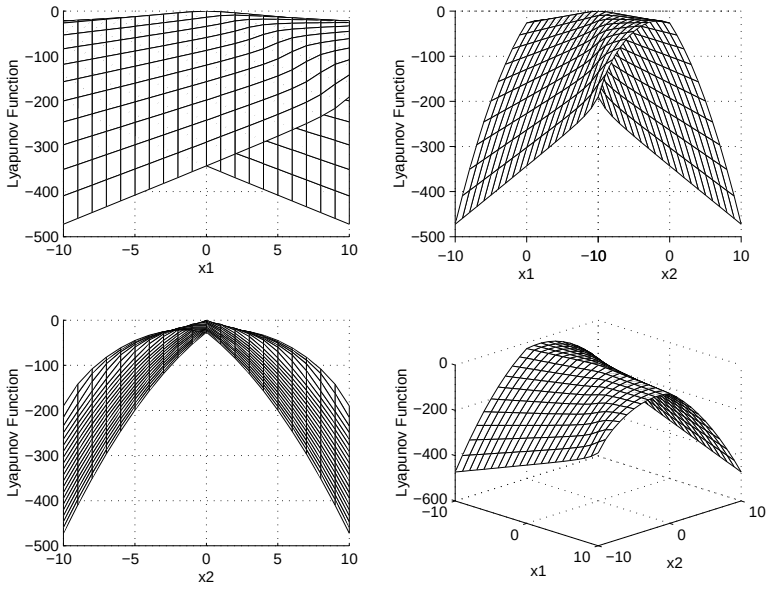


Figure 1: *Lyapunov function when stable* ($a_1 = -1, a_2 = -1, a_3 = -1.8, C = 2, K = 3$)*, front view (top-left),* 45*-degree view (top-right), side-view (bottom-left) and isometric*

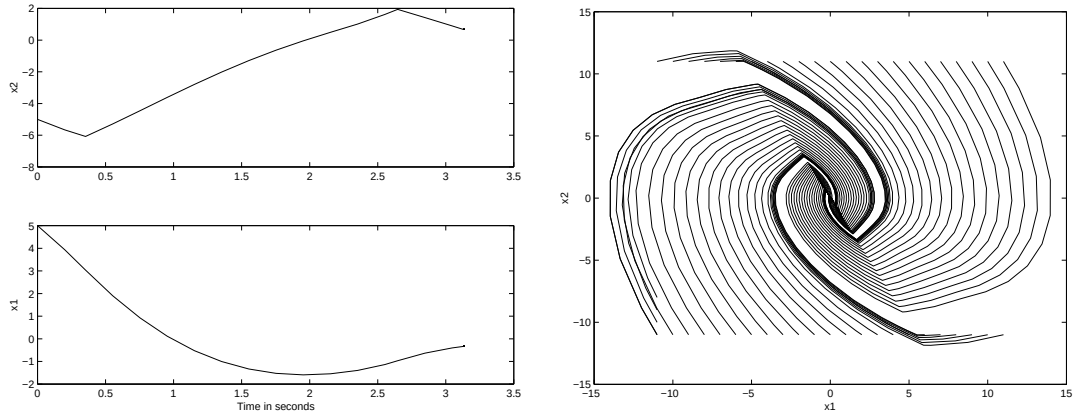


Figure 2: *(a) State variables (initial condition is* $(x_1, x_2) = (5, -5)$*,* $t_{CPU} = 3.0900\,\text{sec}$*, file size = 125.57 KB) and (b) state plane (initial condition (planned) is* $(x_1, x_2) = \{-11 \leq \mathbb{I} \leq 11\} \times \{-11 \leq \mathbb{I} \leq 11\}$*,* $t_{CPU} = 92.2800\,\text{sec}$*, file size = 1.78 MB) when stable* ($a_1 = -1, a_2 = -1, a_3 = -1.8, C = 2, K = 3$)

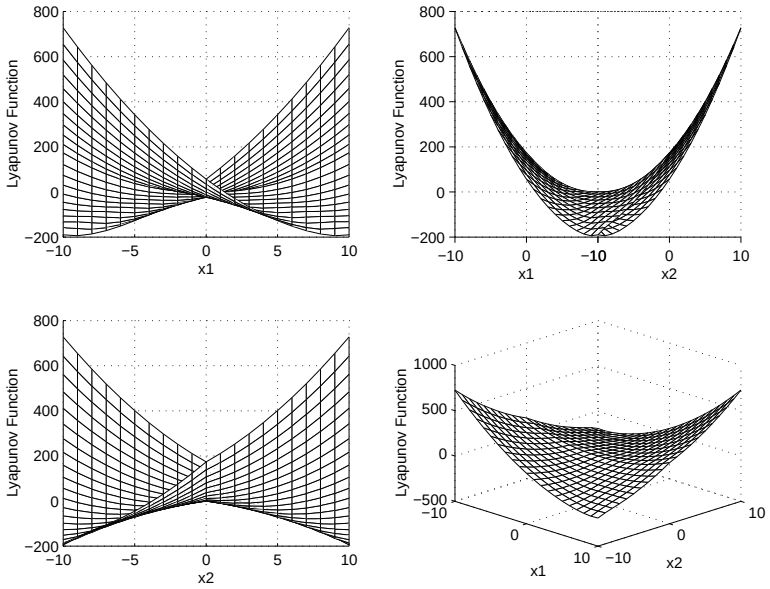
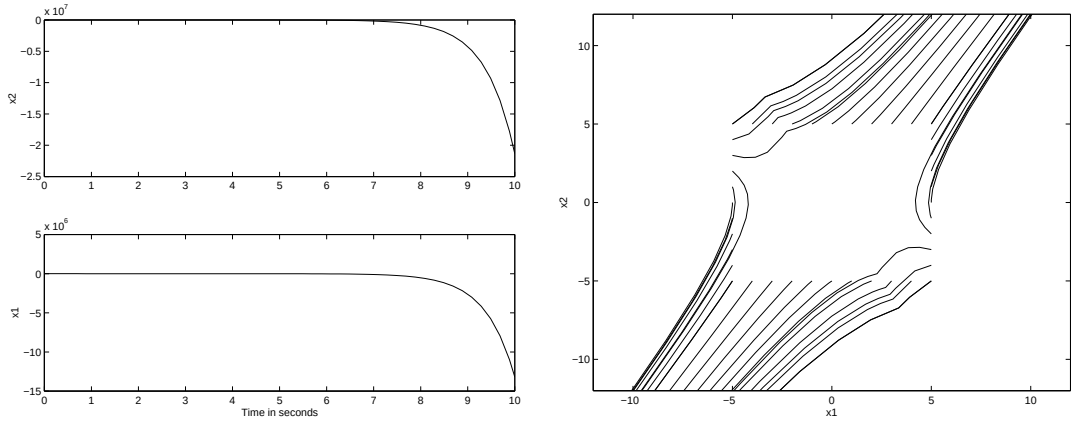


Figure 3: *Lyapunov function when unstable ($a_1 = 1, a_2 = 1, a_3 = -1.8, C = 2, K = 3$), front view (top-left), 45-degree view (top-right), side-view (bottom-left) and isometric*



(a) (b)

Figure 4: *(a) State variables (initial condition is $(x_1, x_2) = (5, -5)$, $t_{simulation} = 10\,\text{sec}, t_{CPU,average} = 0.0200\,\text{sec}$, file size = 9.34 KB) and (b) state plane (initial condition is $(x_1, x_2) = \{-5 \leq \mathbb{I} \leq 5\} \times \{-5 \leq \mathbb{I} \leq 5\}$, $t_{simulation} = 10\,\text{sec}, t_{CPU} = 0.0150\,\text{sec}$, file size = 9.39 KB) when stable ($a_1 = 1, a_2 = 1, a_3 = -1.8, C = 2, K = 3$)*

Figure 5 and Figure 6 show the responses of a stable system but with a different set of parameters to those of the system whose result was shown in Figure 1 and Figure 2 (here $a_1 = -1, a_2 = -1, a_3 = -1.8, C = 1, K = 2$).

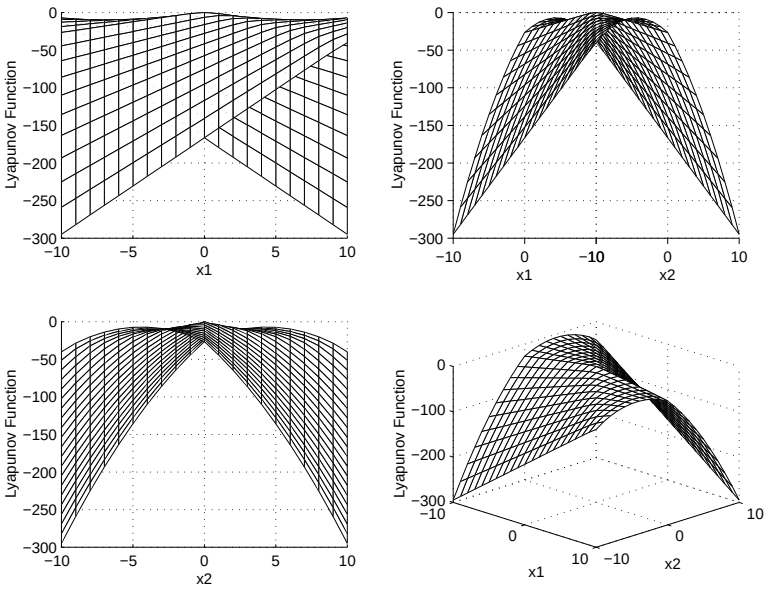
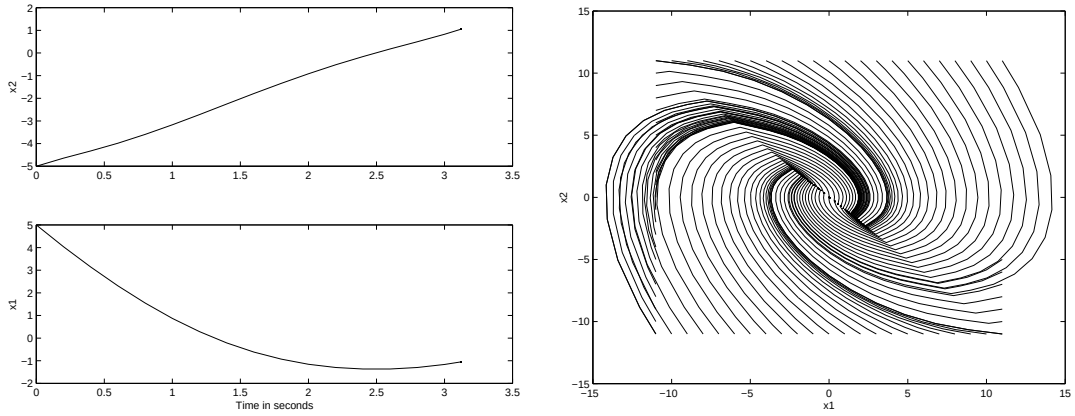


Figure 5: *Lyapunov function when stable ($a_1 = -1, a_2 = -1, a_3 = -1.8, C = 1, K = 2$), front view (top-left),* 45*-degree view (top-right), side-view (bottom-left) and isometric*



(a) (b)

Figure 6: *(a) State variables (initial condition is* $(x_1, x_2) = (5, -5)$*,* $t_{CPU} = 12.1000\,\text{sec}$*, file size = 518.81 KB) and (b) state plane (initial condition (planned) is* $(x_1, x_2) = \{-11 \leq \text{Ⅱ} \leq 11\} \times \{-11 \leq \text{Ⅱ} \leq 11\}$*,* $t_{CPU} = 78.2800\,\text{sec}$*, file size = 1.62 MB) when stable (*$a_1 = -1, a_2 = -1, a_3 = -1.8, C = 1, K = 2$*)*

Related to Problem 2 consider the same problem

$$\dot{x_1} = x_2 \tag{16}$$
$$\dot{x_2} = a_1 x_1 + a_2 x_2 + a_3 \tan^{-1}(x_1 + x_2) + u \tag{17}$$

but this time none of the parameters a_1, a_2, a_3 is known. In order to design an adaptive control scheme for this system let the Lyapunov function be

$$V = \frac{1}{2}s^2 + \frac{1}{2}\begin{bmatrix} \tilde{a_1} & \tilde{a_2} & \tilde{a_3} \end{bmatrix} \Gamma^{-1} \begin{bmatrix} \tilde{a_1} \\ \tilde{a_2} \\ \tilde{a_3} \end{bmatrix}, \tag{18}$$

where $\tilde{a_1} = a_1 - \bar{a_1}$, $\tilde{a_2} = a_2 - \bar{a_2}$ and $\tilde{a_3} = a_3 - \bar{a_3}$.

Let $\tilde{a} = \begin{bmatrix} \tilde{a_1} & \tilde{a_2} & \tilde{a_3} \end{bmatrix}^T$ be the actual parameters which are unknown. If we let $s = C\,x_1 + x_2$ then

$$\dot{V} = s\dot{s} + \tilde{a}^T \Gamma^{-1} \dot{\tilde{a}} \tag{19}$$
$$= \bar{a_1} C\,x_1^2 + (1 + \bar{a_2})\,x_2^2 + (\bar{a_1} + C + \bar{a_2} C)\,x_1 x_2 + \tag{20}$$
$$\bar{a_3}\,(C\,x_1 + x_2)\tan^{-1}(x_1 + x_2) - K(C\,x_1 + x_2)\mathrm{sgn}(\mathrm{C\,x_1 + x_2}) + \tag{21}$$
$$(C\,x_1 + x_2)\,(a_1 - \bar{a_1})\,x_1 + (C\,x_1 + x_2)\,(a_2 - \bar{a_2})\,x_2 + \tag{22}$$
$$(C\,x_1 + x_2)\,(a_3 - \bar{a_3})\,x_2 + \tilde{a}^T \Gamma^{-1} \dot{\tilde{a}} \tag{23}$$

For simplicity let $\Gamma = \begin{bmatrix} \gamma_1 & 0 & 0 \\ 0 & \gamma_2 & 0 \\ 0 & 0 & \gamma_3 \end{bmatrix}$ so that

$$\tilde{a}^T \begin{bmatrix} (C\,x_1 + x_2)\,x_1 \\ (C\,x_1 + x_2)\,x_2 \\ (C\,x_1 + x_2)\,x_3 \end{bmatrix} + \tilde{a}^T \Gamma^{-1} \dot{\tilde{a}} = 0. \tag{24}$$

Then

$$\dot{\tilde{a}} = -\Gamma \begin{bmatrix} (C\,x_1 + x_2)\,x_1 \\ (C\,x_1 + x_2)\,x_2 \\ (C\,x_1 + x_2)\tan^{-1}(x_1 + x_2) \end{bmatrix}, \tag{25}$$

that is

$$\begin{bmatrix} \dot{a_1} - \dot{\bar{a_1}} \\ \dot{a_2} - \dot{\bar{a_2}} \\ \dot{a_3} - \dot{\bar{a_3}} \end{bmatrix} = -\Gamma \begin{bmatrix} (C\,x_1 + x_2)\,x_1 \\ (C\,x_1 + x_2)\,x_2 \\ (C\,x_1 + x_2)\tan^{-1}(x_1 + x_2) \end{bmatrix}. \tag{26}$$

Because a_1, a_2, a_3 are constants $\dot{a_1} = \dot{a_2} = \dot{a_3} = 0$, therefore the adaptive schemes are

$$\dot{\bar{a_1}} = \gamma_1\,(C\,x_1 + x_2)\,x_1 \tag{27}$$
$$\dot{\bar{a_2}} = \gamma_2\,(C\,x_1 + x_2)\,x_2 \tag{28}$$
$$\dot{\bar{a_3}} = \gamma_3\,(C\,x_1 + x_2)\tan^{-1}(x_1 + x_2), \tag{29}$$

the initial estimates were set to be $\bar{a}_1(0) = \bar{a}_2(0) = \bar{a}_3(0) = 0$.

In order to guarantee that Equation 27 to Equation 29 converge the conditions

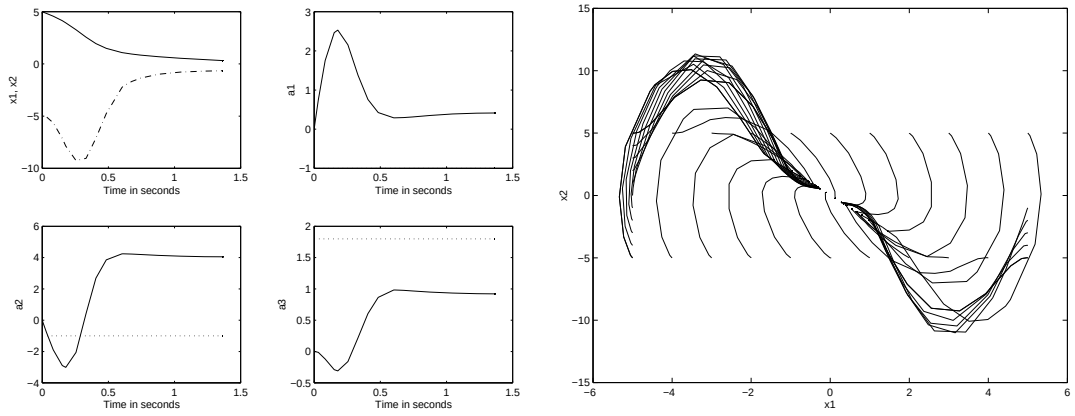
$$\gamma_1 < 0 \tag{30}$$
$$\gamma_2 < 0 \tag{31}$$

must be true (but in doing simulation when $\gamma_1, \gamma_2 < 0$ there were singularities in the system and unable to simulate the system.)

The control scheme used in simulations is

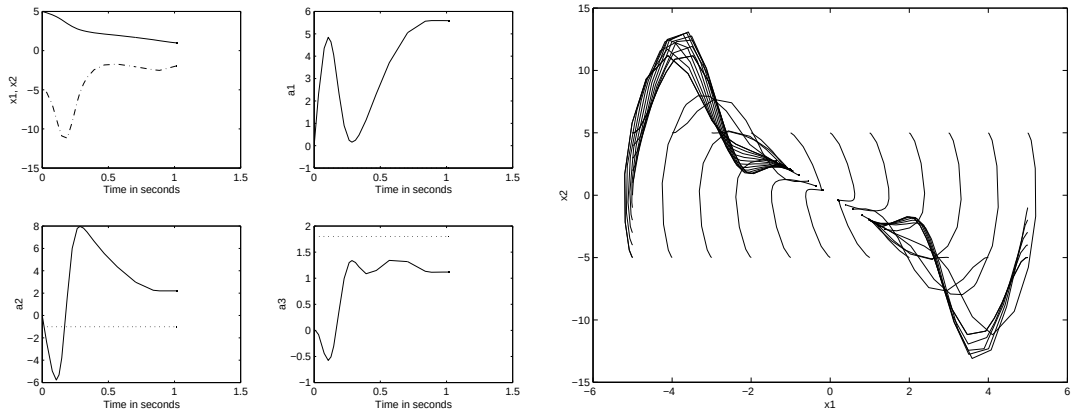
$$u = -x_2 - \bar{a}_1 x_1 - \bar{a}_2 x_2 - \bar{a}_3 \tan^{-1}(x_1 + x_2) - K \operatorname{sgn}(\mathrm{C}\,\mathrm{x}_1 + \mathrm{x}_2) \tag{32}$$

Figure 7 shows the results when $\gamma_1 = \gamma_2 = \gamma_3 = 1$ while Figure 8 shows the results when $\gamma_1 = \gamma_2 = \gamma_3 = 3$.



(a) (b)

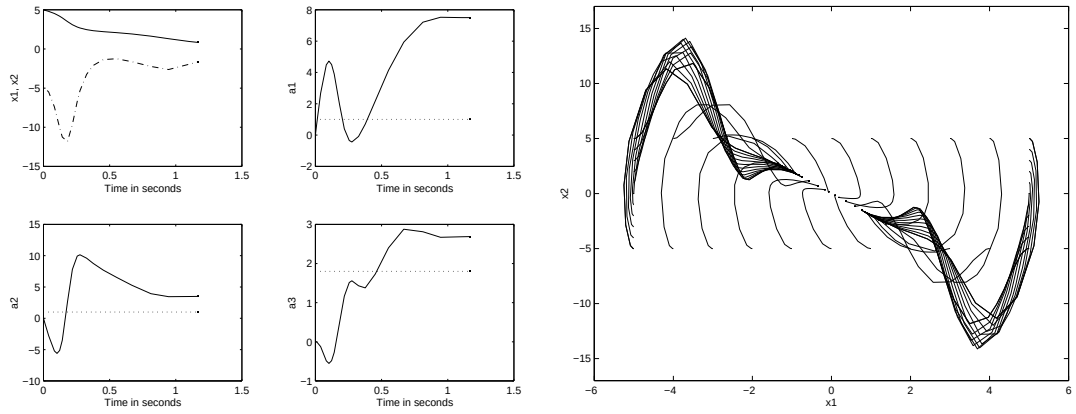
Figure 7: *(a) State variables (x_1, x_2) and estimated parameters $(\bar{a_1}, \bar{a_2}, \bar{a_3})$ (initial condition is $(x_1, x_2) = (5, -5)$, $t_{CPU} = 3.7900\,\mathrm{sec}$, file size = 272.46 KB) and (b) state plane (initial condition (planned) is $(x_1, x_2) = \{-5 \leq \mathrm{I\!I} \leq 5\} \times \{-5 \leq \mathrm{I\!I} \leq 5\}$, $t_{CPU} = 72.9200\,\mathrm{sec}$, file size = 563.20 KB) when stable ($a_1 = -1, a_2 = -1, a_3 = -1.8, C = 2, K = 3, \gamma_1 = \gamma_2 = \gamma_3 = 1$)*



(a) (b)

Figure 8: *(a) State variables (x_1, x_2) and estimated parameters $(\bar{a_1}, \bar{a_2}, \bar{a_3})$ (initial condition is $(x_1, x_2) = (5, -5)$, $t_{CPU} = 4.4500\,\mathrm{sec}$, file size = 335.20 KB) and (b) state plane (initial condition (planned) is $(x_1, x_2) = \{-5 \leq \mathrm{I\!I} \leq 5\} \times \{-5 \leq \mathrm{I\!I} \leq 5\}$, $t_{CPU} = 56.9100\,\mathrm{sec}$, file size = 517.48 KB) when stable ($a_1 = -1, a_2 = -1, a_3 = -1.8, C = 2, K = 3, \gamma_1 = \gamma_2 = \gamma_3 = 3$)*

Figure 9 shows how this control scheme works for the case which was previously considered unstable (see Figure 4 for comparison.)



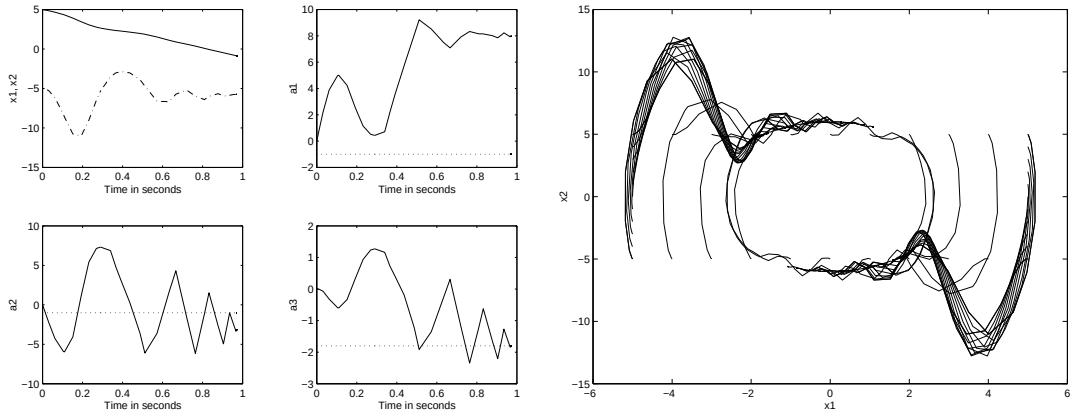
(a) (b)

Figure 9: *(a) State variables* (x_1, x_2) *and estimated parameters* $(\bar{a_1}, \bar{a_2}, \bar{a_3})$ *(initial condition is* $(x_1, x_2) = (5, -5)$*,* $t_{CPU} = 3.7700\,\text{sec}$*, file size = 319.02 KB) and (b) state plane (initial condition (planned) is* $(x_1, x_2) = \{-5 \leq \mathbb{I} \leq 5\} \times \{-5 \leq \mathbb{I} \leq 5\}$*,* $t_{CPU} = 63.4200\,\text{sec}$*, file size = 571.84 KB) when stable (*$a_1 = 1, a_2 = 1, a_3 = -1.8, C = 2, K = 3, \gamma_1 = \gamma_2 = \gamma_3 = 3$*)*

In order to better the parameter estimation try using more than one hyperplane. For example, let

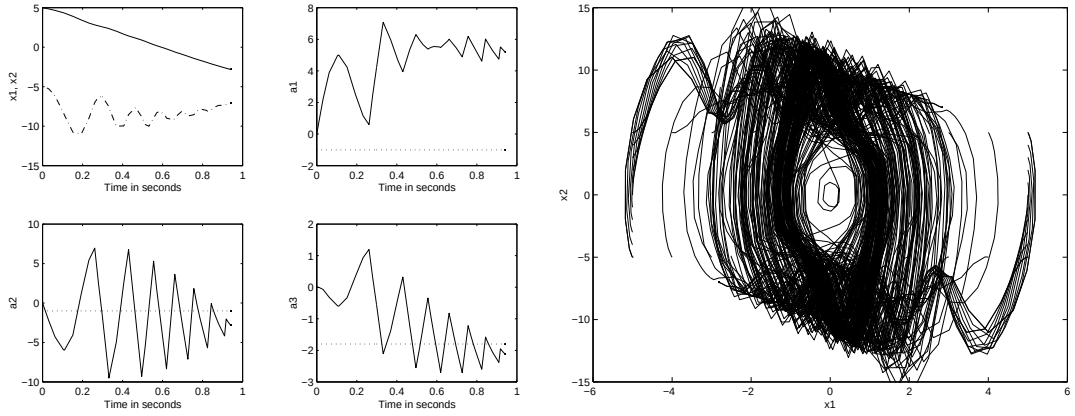
$$s = \begin{cases} C\,x_1 + x_2, & x_1^2 + x_2^2 > r^2 \\ C\,x_1, & x_1^2 + x_2^2 = r^2 \\ C\,x_1 - x_2, & x_1^2 + x_2^2 < r^2 \end{cases} . \tag{33}$$

Examples of the result are shown in Figure 10 and Figure 11.



(a) (b)

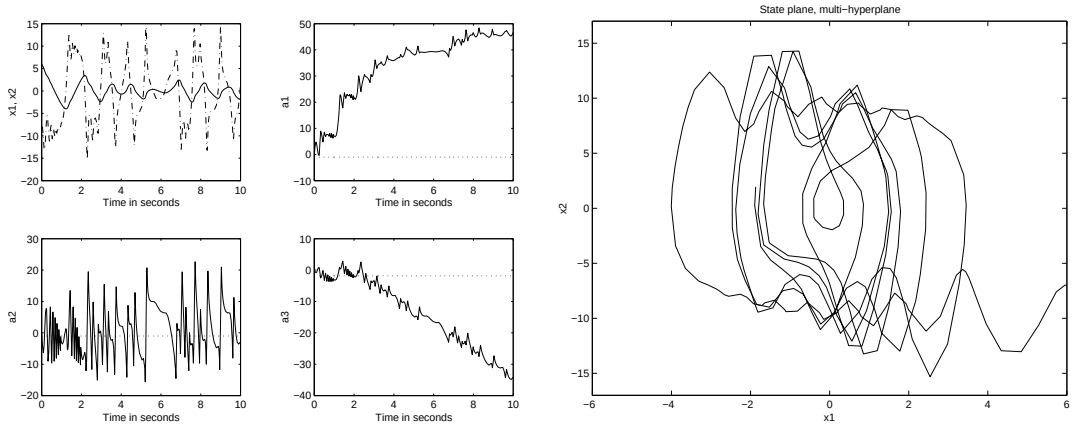
Figure 10: *(a) State variables and (b) state plane, with more than one hyperplane* ($a_1 = -1$, $a_2 = -1$, $a_3 = -1.8$, $C = 2$, $K = 3$, $r = 6$, $\gamma_1 = \gamma_2 = \gamma_3 = 3$*)* $t_{CPU} = 2.6800\,\text{sec}$ *for (a) and* $t_{CPU} = 39.8900\,\text{sec}$ *for (b), initial conditions used were* $(x_1, x_2) = \{5, -5\} \times \{-5 \leq \text{Ⅱ} \leq 5\}$



(a) (b)

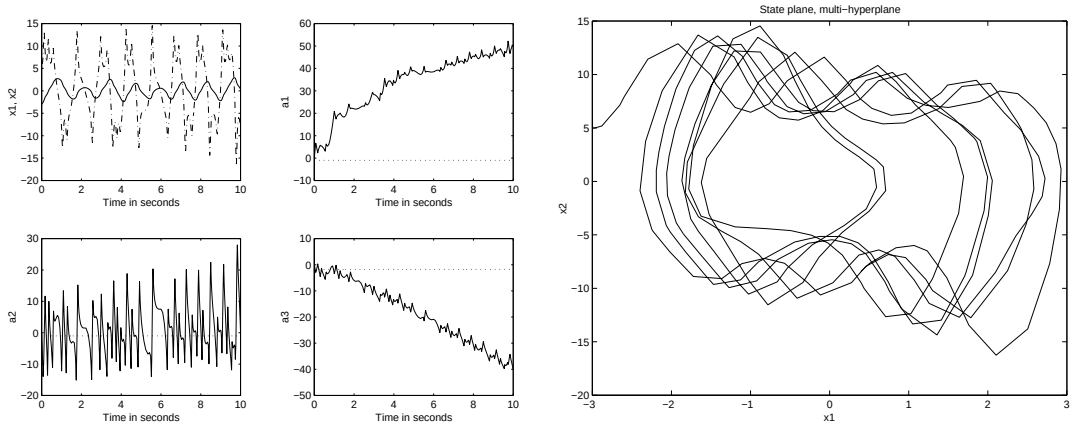
Figure 11: *(a) State variables and (b) state plane, with more than one hyperplane* ($a_1 = -1$, $a_2 = -1$, $a_3 = -1.8$, $C = 2$, $K = 3$, $r = 9$, $\gamma_1 = \gamma_2 = \gamma_3 = 3$*)* $t_{CPU} = 5.4300\,\text{sec}$ *for (a), initial conditions used were* $(x_1, x_2) = \{5, -5\} \times \{-5 \leq \text{Ⅱ} \leq 5\}$

Or look at the response for ony one initial condition at a time in Figure 12



(a) (b)

Figure 12: *(a) State variables and (b) state plane, multi-hyperplane* ($a_1 = -1$, $a_2 = -1$, $a_3 = -1.8$, $C = 2$, $K = 3$, $r^2 = 90$, $\gamma_1 = \gamma_2 = \gamma_3 = 3$*)* $t_{CPU} = 3.7400\,\text{sec}$ *initial conditions used were* $(x_1, x_2) = (7, -8)$



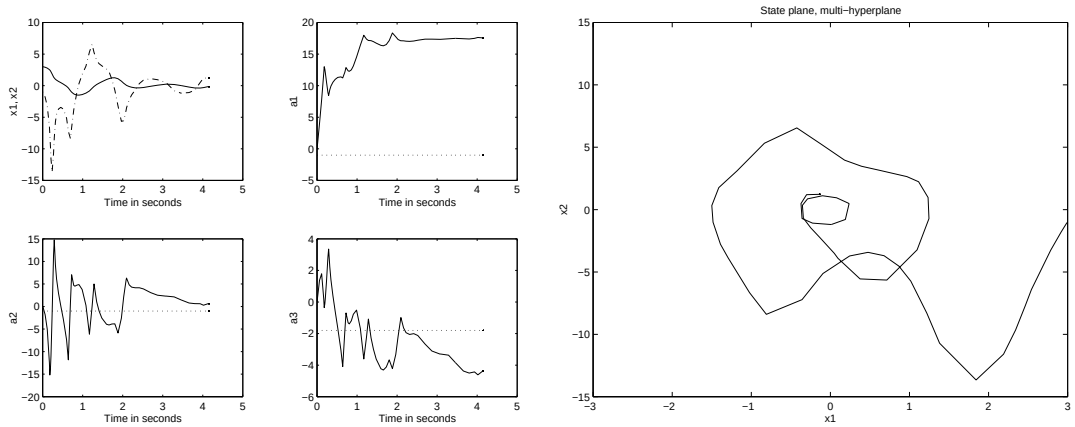
(a) (b)

Figure 13: *(a) State variables and (b) state plane, multi-hyperplane* ($a_1 = -1$, $a_2 = -1$, $a_3 = -1.8$, $C = 2$, $K = 3$, $r^2 = 90$, $\gamma_1 = \gamma_2 = \gamma_3 = 3$*) initial conditions used were* $(x_1, x_2) = (5, -3)$

Let

$$s = \begin{cases} C\,x_1 + x_2, & x_1^2 + x_2^2 > e^{-t}r^2 \\ C\,x_1, & x_1^2 + x_2^2 = e^{-t}r^2 \\ C\,x_1 - x_2, & x_1^2 + x_2^2 < e^{-t}r^2 \end{cases}, \tag{34}$$

simulated and obtained the result as shown in Figure 14.



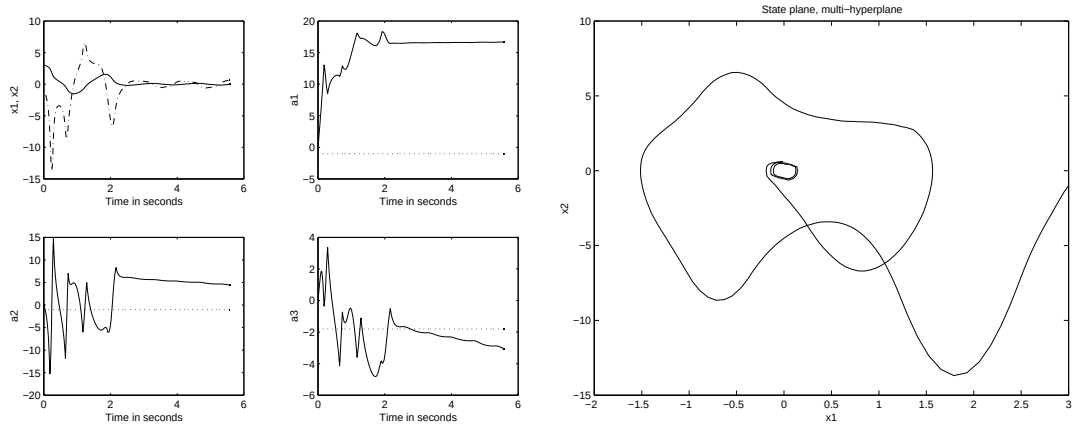
(a) (b)

Figure 14: *(a) State variables and (b) state plane, multi-hyperplane* ($a_1 = -1$, $a_2 = -1$, $a_3 = -1.8$, $C = 2$, $K = 3$, $r^2 = 100$, $\gamma_1 = \gamma_2 = \gamma_3 = 3$) $t_{CPU} = 13.4700\,\text{sec}$*, hyperplanes switching is related to* $e^{-t}\left(x_1^2 + x_2^2\right)$ *initial conditions used were* $(x_1, x_2) = (3, -1)$

Then do the same as above but with a dither signal

$$u = -x_2 - \bar{a}_1 x_1 - \bar{a}_2 x_2 - \bar{a}_3 \tan^{-1}(x_1 + x_2) - K \operatorname{sgn}(\mathrm{C\,x_1 + x_2}) + \mathrm{v}, \tag{35}$$

where v is a zero-mean random signal with a small variance σ_v. The result is shown in Figue 15.



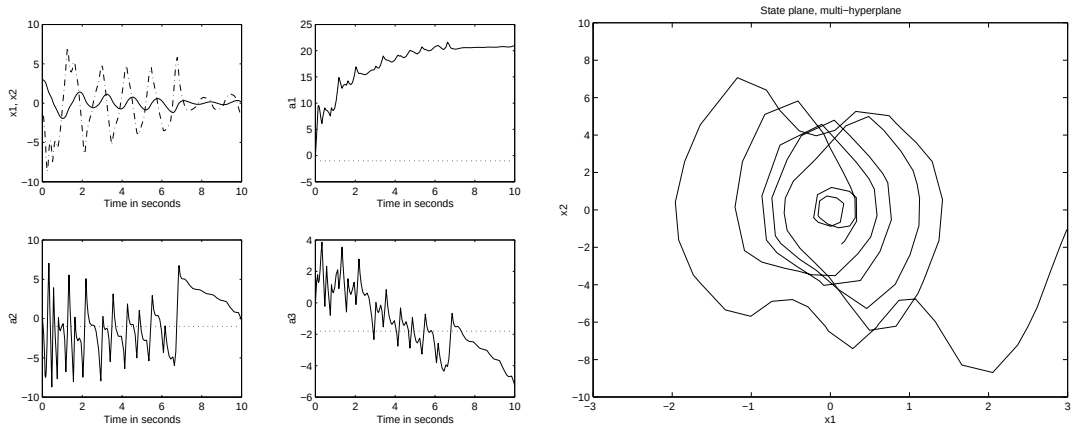
(a) (b)

Figure 15: *(a) State variables and (b) state plane, multi-hyperplane* ($a_1 = -1$, $a_2 = -1$, $a_3 = -1.8$, $C = 2$, $K = 3$, $r^2 = 100$, $\gamma_1 = \gamma_2 = \gamma_3 = 3$) $t_{CPU} = 12.5300\,\mathrm{sec}$, $\sigma_v = 1$, *dither signal has sampling time* 0.01 *seconds, hyperplanes switching is related to* $e^{-t}(x_1^2 + x_2^2)$ *initial conditions used were* $(x_1, x_2) = (3, -1)$

Let

$$s = \begin{cases} C\,x_1 + x_2, & x_1^2 + x_2^2 > \frac{1}{(t+2)} r^2 \\ C\,x_1, & x_1^2 + x_2^2 = \frac{1}{(t+2)} r^2 \\ C\,x_1 - x_2, & x_1^2 + x_2^2 < \frac{1}{(t+2)} r^2 \end{cases}, \tag{36}$$

simulated and obtained the result as shown in Figure 16.



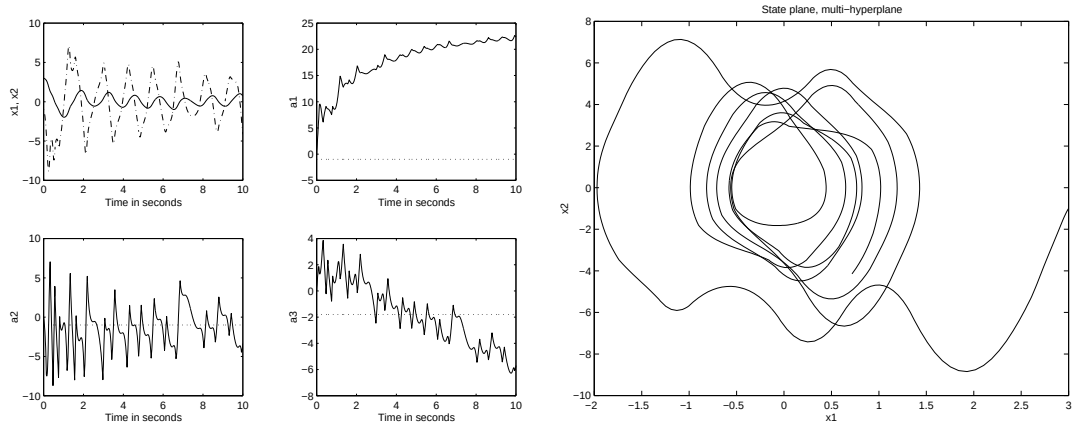
(a) (b)

Figure 16: *(a) State variables and (b) state plane, multi-hyperplane* ($a_1 = -1$, $a_2 = -1$, $a_3 = -1.8$, $C = 2$, $K = 3$, $r^2 = 100$, $\gamma_1 = \gamma_2 = \gamma_3 = 3$) $t_{CPU} = 0.3900\,\text{sec}$, *hyperplanes switching is related to* $\frac{1}{(t+2)}\left(x_1^2 + x_2^2\right)$ *initial conditions used were* $(x_1, x_2) = (3, -1)$

When a dither signal is added to the input

$$u = -x_2 - \bar{a}_1 x_1 - \bar{a}_2 x_2 - \bar{a}_3 \tan^{-1}(x_1 + x_2) - K \operatorname{sgn}(\mathrm{C\,x_1 + x_2}) + \mathrm{v}, \tag{37}$$

where v is a zero-mean random signal with a small variance σ_v. Figure 17 shows the result from simulation.



(a) (b)

Figure 17: *(a) State variables and (b) state plane, multi-hyperplane* ($a_1 = -1$, $a_2 = -1$, $a_3 = -1.8$, $C = 2$, $K = 3$, $r^2 = 100$, $\gamma_1 = \gamma_2 = \gamma_3 = 3$) $t_{CPU} = 1.9600\,\mathrm{sec}$, *hyperplanes switching is related to* $\frac{1}{(t+2)}(x_1^2 + x_2^2)$, $\sigma_v = 0.02$, *sampling time of the dither signal is* 0.01 *seconds, initial conditions used were* $(x_1, x_2) = (3, -1)$

Trajectory control of a synchronous machine

This section studies tracking control for a two dimensional model of a synchronous generator (this is continued from my previous work [Tiy98].)

Suppose that the objective of control for a synchronous generator is such that its states will track a circular orbit around the origin. This may seem to be unrealistic since the desired final state in controlling a synchronous generator is to lead the states toward the horizontal axis in the state plane where

$$\delta = \delta_{design}, \quad \dot{\delta} = 0.$$

A simplified version of the same problem is to make the trajectory go toward the origin where

$$\delta \quad = \dot{\delta} \quad = 0.$$

However the idea is that if we can make the trajectory track an orbit and if we keep reducing the radius of the orbit with each revolution the trajectory will eventually move to a close proximity of the origin. This approach can be justified if one consider the fact that instead of having an equilibrium point at the origin, a synchronous machine has equilibrium points at

$$(x_1, x_2) = \left(\sin^{-1}\frac{P_m H}{P_e}, 0\right). \tag{38}$$

This means that a typical approach of stabilizing the origin is no longer applicable here.

The model of a synchronous generator which will be considered here is given as Equation 45 in the Technical Report #6 [Tiy98] while the state equations are given as Equation 46 and 47, while the control is derived in Equation 93 to 101 and then The input is then

$$u = -K\tanh\left(\gamma\mathcal{F}\mathcal{F}_x^T\mathcal{V}\right), \tag{39}$$

where $\mathcal{V}$ is a vector satisfying $\mathcal{V}u$ and the larger the $\gamma \in \mathcal{R}$ is the closer the approximation of $\mathrm{sgn}(\cdot)$ by $\tanh(\cdot)$. The input u is actually the mechanical input power P_m where a positive value signifies an input and a negative value a load.

The simulation result in Technical Report #6 has one problem, that is the controlled trajectory stops on the desired orbit after having reached it. This may be due to the use of a sign function.

The following simulation uses a hyperbolic tangent in place of a sign function in order to compare the result.

Here instead of Equation 108 in the Technical Report #6 the control input becomes

$$\begin{aligned} u &= -K\tanh\left(g\mathcal{F}\mathcal{F}_x^T\mathcal{V}_1\right) \\ &= -K\tanh\left(\frac{1}{H}\left(x_1^2(t)+x_2^2(t)-r^2\right)\begin{bmatrix} 2x_1(t) & 2x_2(t)\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}\right) \\ &= -K\tanh\left(\frac{2}{H}\left(x_1(t)+x_2(t)\right)\left(x_1^2(t)+x_2^2(t)-r^2\right)\right), \end{aligned} \tag{40}$$

where $\mathcal{V}_1$ is a unit vector and here $g = \frac{1}{H}$.

Figure 18 shows result when

$$K = 3, \quad H = 0.8, \quad P_e = 1.3, \quad r = 2.$$

The initial condition is Initial Condition Set 1.

Initial Condition Set 1 `n = 0;`

```
for i = -5 to 5 steps 2,
  n = n + 1; x1(0) = i and  x2(0) = i are the n_th initial condition;
  n = n + 1; x1(0) = i and  x2(0) = -i are the n_th initial condition;
endfor;
```

Then Figure 19 shows the result when K is increased ten times. The parameter set becomes

$$K = 30, \quad H = 0.8, \quad P_e = 1.3, \quad r = 2,$$

while the initial condition is still Initial Condition Set 1.

The result shows an improvement to the result in Technical Report #6 because the trajectory is now moving along the orbital curve. But the trajectory does not complete the orbit. Instead it stops when the points $(\pm 2, 0)$ are reached. When K is increased 10 times (Figure 19) the response is still similar to that with a smaller K except that there is less oscillation around the points $(\pm 2, 0)$.

Suppose that the design objective is to bring the speed to close to the synchronous speed and to bring the power angle to zero with the tolerance of 0.02 radian, in other words

$$\delta = x_1 \leq 0.02, \quad \dot{\delta} = x_2 = 0,$$

the power angle being approximately the angle δ (Equation 14 in the Technical Repoart #6.)

Let the parameters of the control system be

$$r = 0.01, \quad K = 10, \quad \gamma = 50.$$

The result of the simulation is shown in Figure 20. All the design criteria is met.

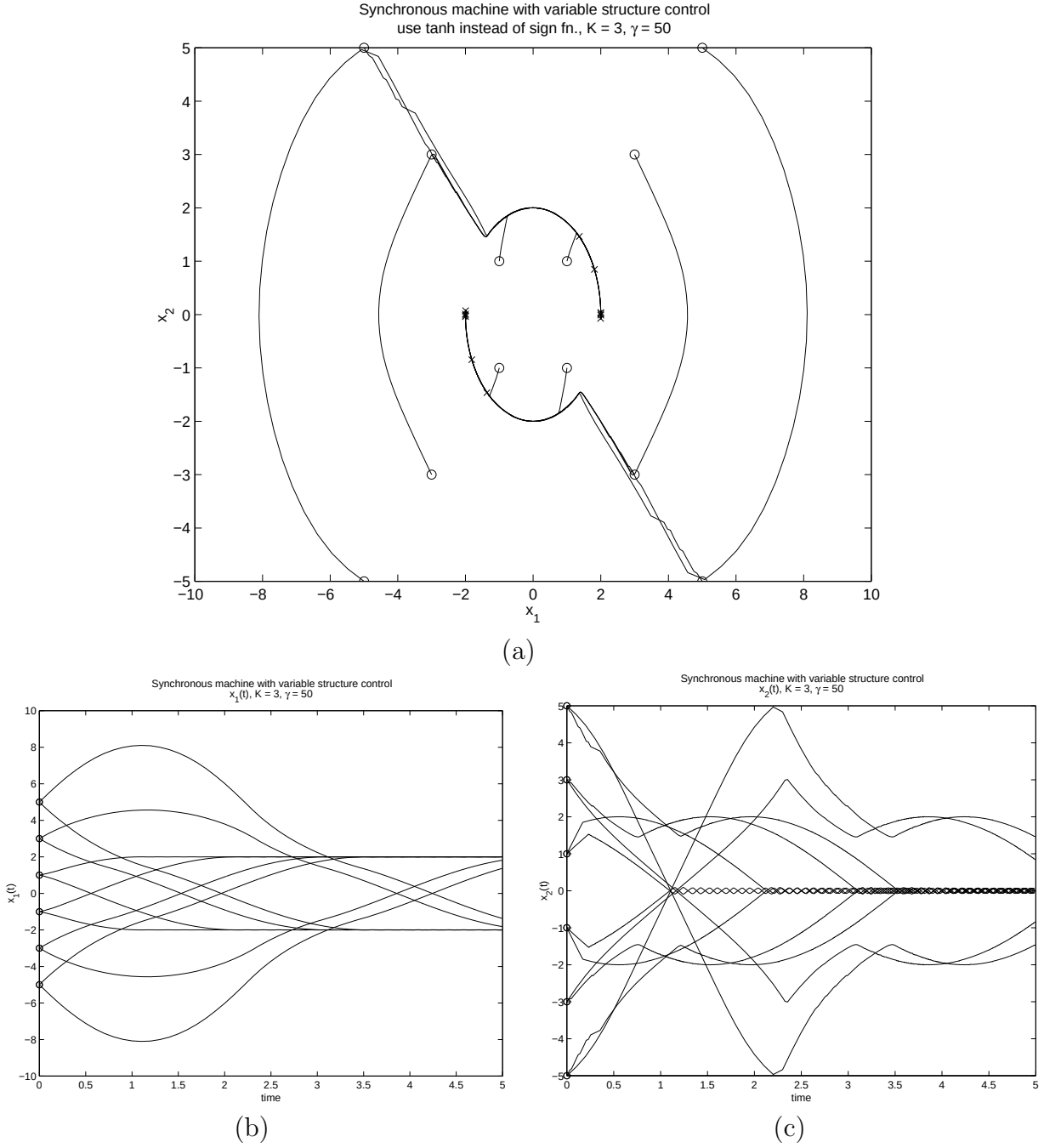
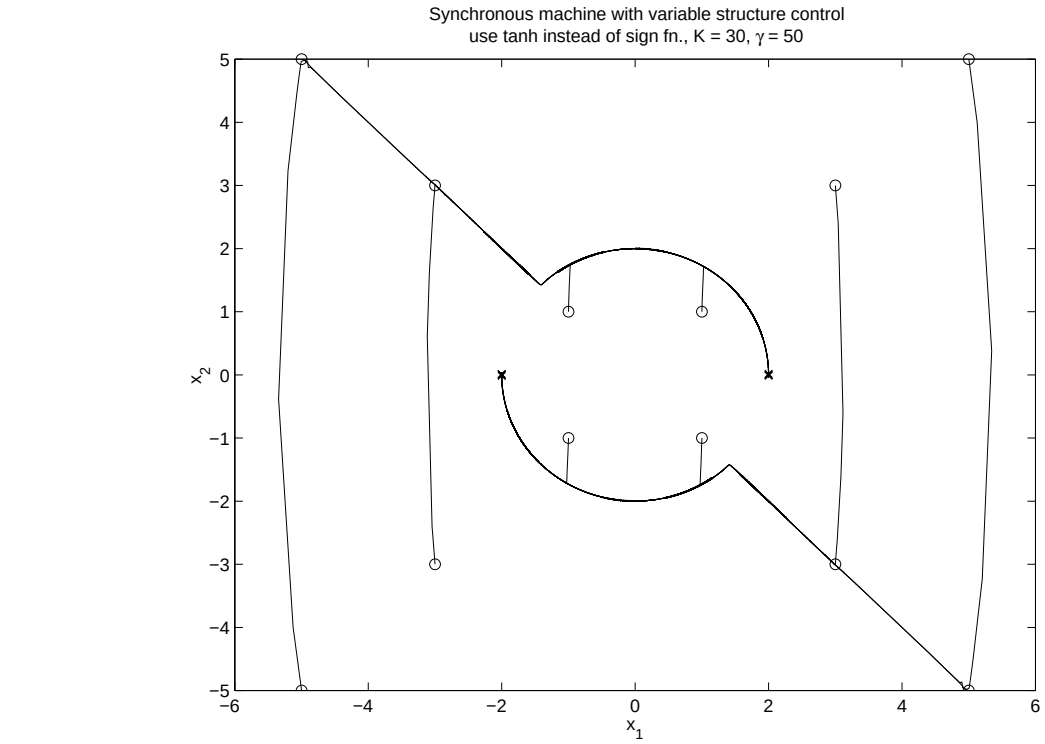
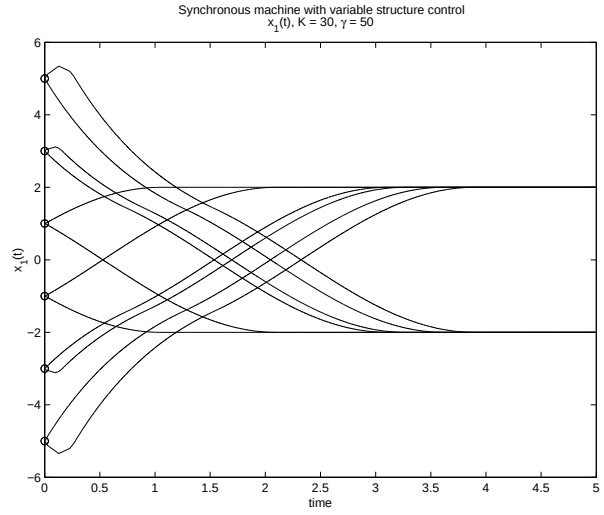


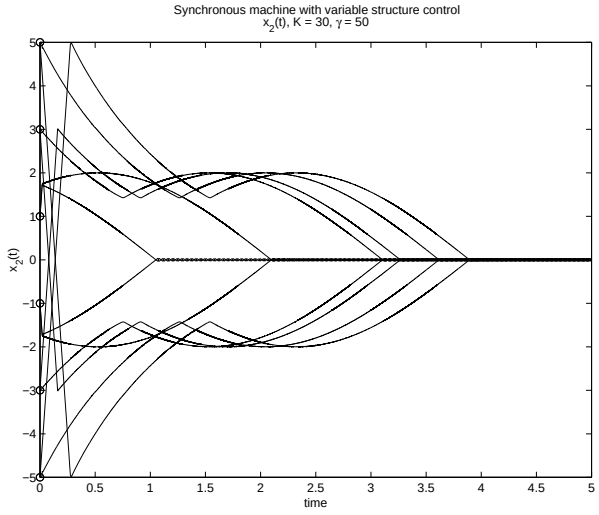
Figure 18: *Synchronous machine with a* tanh *variable structure control (a) state plane, (b)* $x_1(t)$*, (c)* $x_2(t)$



(a)



(b)



(c)

Figure 19: *Synchronous machine with a* tanh *variable structure control (a) state plane, (b)* $x_1(t)$*, (c)* $x_2(t)$

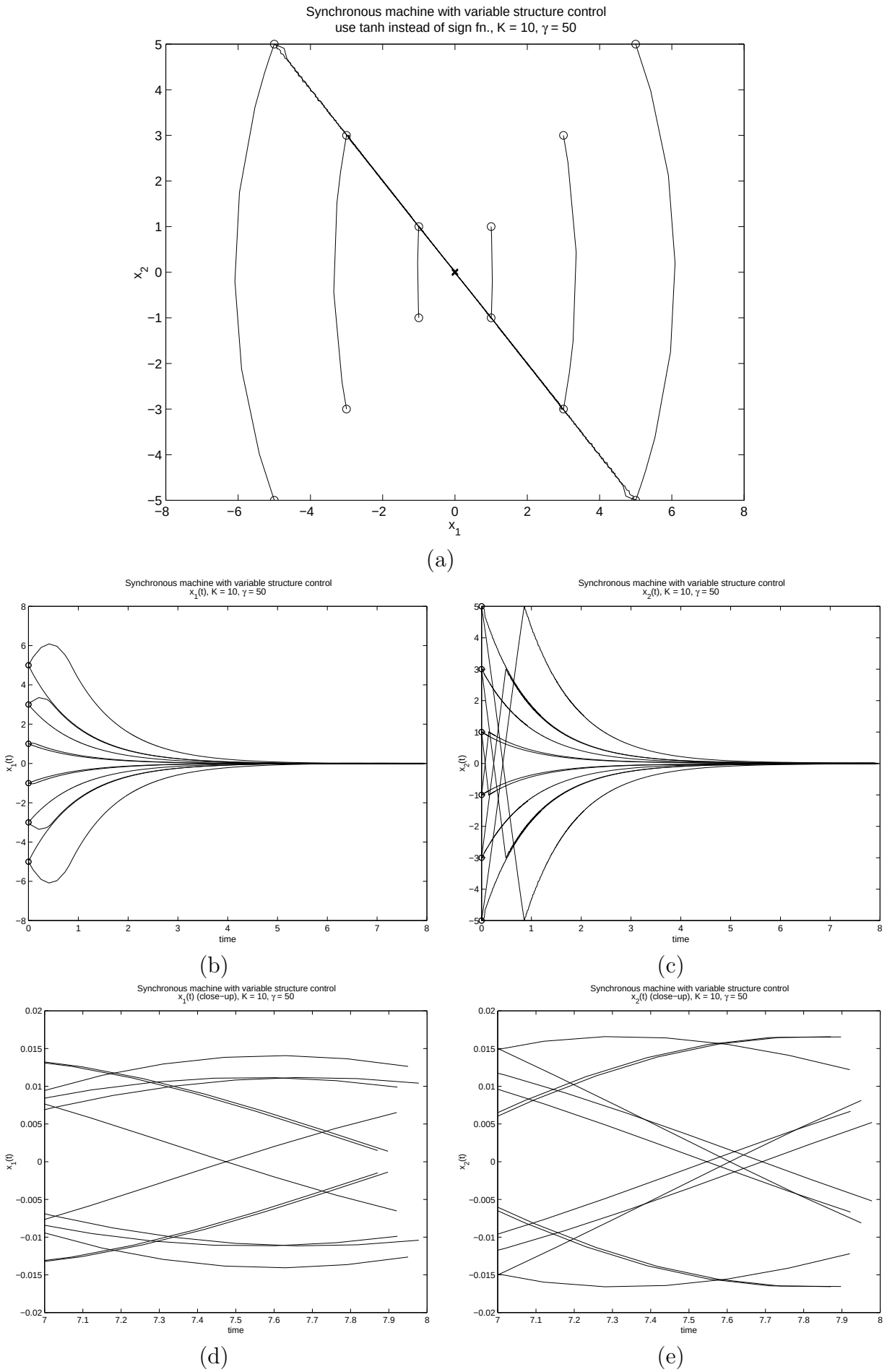


Figure 20: *Synchronous machine with a* tanh *variable structure control (a) state plane, (b)* $x_1(t)$*, (c)* $x_2(t)$*, (d) close-up of (b), and (e) close-up of (c)*

If the desired angle δ is specified as δ_s, then one chooses $\mathcal{F}$ as

$$\mathcal{F} = (x_1(t) - \delta_s)^2 + x_2^2(t) - r^2. \tag{41}$$

Then let the lyapunov function be

$$V = \mathcal{F}^2 = \left((x_1(t) - \delta_s)^2 + x_2^2(t) - r^2\right)^2, \tag{42}$$

which has a derivative

$$\begin{aligned} \dot{V} &= 2\left((x_1(t) - \delta_s)^2 + x_2^2(t) - r^2\right)\left(2\left(x_1(t) - \delta_s\right)\dot{x}_1(t) + 2x_2(t)\dot{x}_2(t)\right) && (43) \\ &= 4\left((x_1(t) - \delta_s)^2 + x_2^2(t) - r^2\right)\left(\left(x_1(t) - \delta_s\right)x_2(t) + x_2(t)\left(\frac{P_m}{H} - \frac{P_e}{H}\sin\left(x_1(t)\right)\right)\right). && (44) \end{aligned}$$

The control input is then

$$u = -K\tanh\left(\gamma\left(\left((x_1 - \delta_s)^2 + x_2^2 - r^2\right)\left(2\left(x_1 - \delta_s\right) + 2x_2\right)\right)\right). \tag{45}$$

Note that in practice γ depends on the quickness of response of the machine supplying the mechanical input. Thus γ can not be as large as possible. This is also another reason why the approximation of $\mathrm{sgn}(\cdot)$ by $\tanh(\cdot)$ is justified.

Two-machine system

For an unregulated two-machine system the system equation is [AF77]

$$\frac{2H_1}{\omega_R}\frac{d^2\delta_{\Delta 1}}{dt^2} + P_{s12}\delta_{12\Delta} = 0 \tag{46}$$

$$\frac{2H_2}{\omega_R}\frac{d^2\delta_{\Delta 2}}{dt^2} + P_{s21}\delta_{21\Delta} = 0 \tag{47}$$

$$\delta_{12\Delta} + \delta_{21\Delta} = 0 \tag{48}$$

Three-machine system

For an unregulated three-machine system the system equation is [AF77]

$$\frac{2H_1}{\omega_R}\frac{d^2\delta_{\Delta 1}}{dt^2} + P_{s12}\delta_{12\Delta} + P_{s13}\delta_{13\Delta} = 0 \tag{49}$$

$$\frac{2H_2}{\omega_R}\frac{d^2\delta_{\Delta 2}}{dt^2} + P_{s21}\delta_{21\Delta} + P_{s23}\delta_{23\Delta} = 0 \tag{50}$$

$$\frac{2H_3}{\omega_R}\frac{d^2\delta_{\Delta 3}}{dt^2} + P_{s31}\delta_{31\Delta} + P_{s32}\delta_{32\Delta} = 0 \tag{51}$$

$$\delta_{12\Delta} + \delta_{23\Delta} + \delta_{31\Delta} = 0 \tag{52}$$

Four-machine system

For an unregulated four-machine system the system equation is [AF77]

$$\frac{2H_1}{\omega_R}\frac{d^2\delta_{\Delta 1}}{dt^2} + P_{s12}\delta_{12\Delta} + P_{s13}\delta_{13\Delta} + P_{s14}\delta_{14\Delta} = 0 \tag{53}$$

$$\frac{2H_2}{\omega_R}\frac{d^2\delta_{\Delta 2}}{dt^2} + P_{s21}\delta_{21\Delta} + P_{s23}\delta_{23\Delta} + P_{s24}\delta_{24\Delta} = 0 \tag{54}$$

$$\frac{2H_3}{\omega_R}\frac{d^2\delta_{\Delta 3}}{dt^2} + P_{s31}\delta_{31\Delta} + P_{s32}\delta_{32\Delta} + P_{s34}\delta_{34\Delta} = 0 \tag{55}$$

$$\frac{2H_4}{\omega_R}\frac{d^2\delta_{\Delta 4}}{dt^2} + P_{s41}\delta_{41\Delta} + P_{s42}\delta_{42\Delta} + P_{s43}\delta_{43\Delta} = 0 \tag{56}$$

$$\delta_{12\Delta} + \delta_{23\Delta} + \delta_{34\Delta} + \delta_{41\Delta} = 0 \tag{57}$$

Unregulated multimachine system

The electrical power output of machine i in an n-machine system is [AF77]

$$\begin{aligned} P_{ei} &= E_i^2 G_{ii} + \sum_{\substack{j=1 \\ j\neq i}}^{n} E_i E_j Y_{ij} \cos\left(\theta_{ij} - \delta_{ij}\right) \\ &= E_i^2 G_{ii} + \sum_{\substack{j=1 \\ j\neq i}}^{n} E_i E_j \left(B_{ij} \sin\delta_{ij} + G_{ij} \cos\delta_{ij}\right) \end{aligned} \tag{58}$$

where
$$\begin{aligned} \delta_{ij} &= \delta_i - \delta_j \\ E_i &= \text{constant voltage behind transient reactance for machine } i \\ \bar{Y}_{ii} &= G_{ii} + jB_{ii} \quad \text{is a diagonal element of } Y \\ \bar{Y}_{ij} &= G_{ij} + jB_{ij} \quad \text{is an off-diagonal element of } Y \\ Y &= \text{the network short circuit admittance matrix.} \end{aligned}$$

Using the incremental model, we have

$$\begin{aligned} \delta_{ij} &= \delta_{ij0} + \delta_{ij\Delta} \\ \sin\delta_{ij} &= \sin\delta_{ij0}\cos\delta_{ij\Delta} + \cos\delta_{ij0}\sin\delta_{ij\Delta} \\ &\approx \sin\delta_{ij0} + \delta_{ij\Delta}\cos\delta_{ij0} \\ \cos\delta_{ij} &\approx \cos\delta_{ij0} - \delta_{ij\Delta}\sin\delta_{ij0} \end{aligned}$$

and
$$P_{ei\Delta} = \sum_{\substack{j=1 \\ j\neq i}}^{n} E_i E_j \left(B_{ij}\sin\delta_{ij0} + G_{ij}\cos\delta_{ij0}\right)\delta_{ij\Delta}. \tag{59}$$

Thus, for a given initial condition

$$P_{ei\Delta} = \sum_{\substack{j=1 \\ j\neq i}}^{n} P_{sij}\delta_{ij\Delta} \tag{60}$$

where
$$\begin{aligned} P_{sij} &\triangleq \left.\frac{\partial P_{ij}}{\partial\delta_{ij}}\right|_{\delta_{ij0}} \\ &= E_i E_j \left(B_{ij}\sin\delta_{ij0} + G_{ij}\cos\delta_{ij0}\right). \end{aligned} \tag{61}$$

Using the inertial model of the synchronous machines, the set of linearized differential equations become

$$\frac{2H_i}{\omega_R}\frac{d^2\delta_{i\Delta}}{dt^2} + \sum_{\substack{j=1 \\ j\neq i}}^{n} E_i E_j \left(B_{ij}\sin\delta_{ij0} + G_{ij}\cos\delta_{ij0}\right)\delta_{ij\Delta} = 0, \qquad i = 1, 2, \ldots, n \tag{62}$$

or
$$\frac{2H_i}{\omega_R}\frac{d^2\delta_{i\Delta}}{dt^2} + \sum_{\substack{j=1 \\ j\neq i}}^{n} P_{sij}\delta_{ij\Delta} = 0, \qquad i = 1, 2, \ldots, n. \tag{63}$$

Since
$$\sum \delta_{ij} = 0 \tag{64}$$

Equation 63 is a set of $(n-1)$-independent equations. From Equation 63,

$$\frac{d^2\delta_{i\Delta}}{dt^2} + \frac{\omega_R}{2H_i}\sum_{\substack{j=1 \\ j\neq i}}^{n} P_{sij}\delta_{ij\Delta} = 0, \qquad i = 1, 2, \ldots, n. \tag{65}$$

Substracting the n^{th} equation from the i^{th} equation,

$$\frac{d^2\delta_{i\Delta}}{dt^2} - \frac{d^2\delta_{n\Delta}}{dt^2} + \frac{\omega_R}{2H_i}\sum_{\substack{j=1\\j\neq i}}^{n} P_{sij}\delta_{ij\Delta} - \frac{\omega_R}{2H_n}\sum_{\substack{j=1\\j\neq i}}^{n} P_{snj}\delta_{nj\Delta} = 0, \quad i = 1, 2, \ldots, n-1 \tag{66}$$

$$\frac{d^2\delta_{in\Delta}}{dt^2} + \frac{\omega_R}{2H_i}\sum_{\substack{j=1\\j\neq i}}^{n} P_{sij}\delta_{ij\Delta} - \frac{\omega_R}{2H_n}\sum_{\substack{j=1\\j\neq i}}^{n} P_{snj}\delta_{nj\Delta} = 0, \quad i = 1, 2, \ldots, n-1. \tag{67}$$

And since,

$$\delta_{ij\Delta} = \delta_{in\Delta} - \delta_{jn\Delta} \tag{68}$$

$$\frac{d^2\delta_{in\Delta}}{dt^2} + \sum_{j=1}^{n-1}\alpha_{ij}\delta_{jn\Delta} = 0, \qquad i = 1, 2, \ldots, n-1, \tag{69}$$

where α_{ij} depend on the machine inertias and synchronizing power coefficients.

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